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Geometries of the Projective Matrix Space, II*

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We have recently investigated the matrix projective line. Our interest was focused at the Möbius transformations $W(Z)=(ZC+D)^{-1}(ZA+B)$, where A, B, C, D, Z are $n\times n$ complex matrices. We were led to generalize the complex plane, the Riemann sphere and the unit disk. Our first task is to study the spherical, Euclidean, and non-Euclidean circles in terms of points at given distance from some point, and in terms of zeroes of some (Hermitian) quadratic form. The relations between the various circles are discussed. We also study Möbius transformations that carry the unit disk into itself, as well as those that carry (or interchange) Hermitian and unitary matrices. Finally, we introduce the stereographic projection from the generalized Riemann sphere to the Euclidean plane, and show that some spherical circles are mapped onto Euclidean ones. © 1986 Academic Press, Inc.

0. INTRODUCTION

In [2] we introduced the complex matrix projective line $P_1(M_n(\mathbb{C}))$, to simplify the study of the Möbius transformations $W(Z)=(ZC+D)^{-1}(ZA+B)$. In [3] we studied the Euclidean, spherical and non Euclidean metrics on the suitable subspaces of $P_1(M_n(\mathbb{C}))$. In particular, the various circles were introduced in terms of their centers and radii, as well as by their associated Hermitian matrix.

The uniqueness of the centers, the radii, and the Hermitian matrix are our goal in the first part.

The comparison of the various circles is our next task. Namely, suppose a Euclidean circle is given. Under what conditions will it be a spherical circle or a non-Euclidean circle? Furthermore, once a given set is a circle say, both in the Euclidean metric, and the spherical one, what are the relations between the centers and the radii in the two metrics? We study in the second part, the various possible cases. Another application of the first part, is a characterization of the Möbius transformations that map, say, the

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unitary matrices into the Hermitian ones, or, the interior of the unit disk into itself.

The Riemann sphere and the stereographic projection are the last topics in this work.

The definitions and notations used in this paper, are those used in [2] and in [3], unless otherwise specified.

As in [2] and [3], most of the motivation arises from the field of complex numbers $\mathbb{C}$. We do not intend to specify, or compare, our methods and results with those known for $\mathbb{C}$. These will be left out, and can easily be verified by the interested reader. We emphasize this remark, in particular with connection to the comparison of the various circles, and the study of the stereographic projection. We shall therefore always assume $n > 1$.

The points of $P_1(M_n(\mathbb{C}))$ will be denoted by $\mathbf{P}$. In the present paper we also denote by $\mathbf{P}$ $n \times 2n$ matrices. This should not lead to any confusion.

For a $2n \times 2n$ Hermitian matrix $\mathscr{H}$, a $n \times 2n$ matrix $\mathbf{P}$ is a zero of $\mathscr{H}$, whenever $\mathbf{P}\mathscr{H}\mathbf{P}^* = 0$. We also say that $\mathbf{P}$ annihilates $\mathscr{H}$.

The Euclidean plane is the set of the finite points $\mathbf{P}$, corresponding to the $n \times 2n$ matrices $\mathbf{P} = (P \;\; I)$.

The sphere K, is the set of $n \times 2n$ matrices $\mathbf{P}$, so that $\mathbf{P}\mathbf{P}^* = I$. The disk Δ^- is the set of points $\mathbf{P}$ so that $\mathbf{P}\mathscr{J}\mathbf{P}^* < 0$ where $\mathscr{J} = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$. For every matrix $\mathbf{P}$ on K there exists a $n \times 2n$ matrix $\hat{\mathbf{P}}$ such that the $2n \times 2n$ matrix $\mathscr{P} = \left(\begin{smallmatrix} \mathbf{P} \\ \hat{\mathbf{P}} \end{smallmatrix}\right)$ is unitary. Every point $\mathbf{P}$ has a representative in K, that is uniquely determined up to a unitary factor. The subset of matrices $\mathbf{P} = (P_1 \;\; P_2)$ in $P_1(M_n(\mathbb{C}))$, that satisfy $P_2 P_2^* - P_1 P_1^* = I$, is denoted by $\mathscr{J}^-$. Every point $\mathbf{P}$ in Δ^- has a representative in $\mathscr{J}^-$, that is uniquely determined up to a unitary factor. For every matrix $\mathbf{P}$ in $\mathscr{J}^-$ there exists a $n \times 2n$ matrix $\check{\mathbf{P}}$ $\mathscr{P} = \left(\begin{smallmatrix} \check{\mathbf{P}} \\ \mathbf{P} \end{smallmatrix}\right)$, and $\mathscr{P}$ is a $\mathscr{J}$-unitary matrix, that is $\mathscr{P}\mathscr{J}\mathscr{P}^* = \mathscr{J}$.

Further properties of matrices in K, and in $\mathscr{J}^-$, and of unitary and $\mathscr{J}$-unitary matrices that we shall use freely, can be found in greater detail in [3].

1. Uniqueness of the Center, the Radius, and the Associated Hermitian Matrix

a. *The Euclidean Circles*

The Euclidean circle $\gamma_e(\mathbf{P}_0, r)$ is defined to be the set of points $\mathbf{P} = (P \;\; I)$ such that $P = P_0 + rU$, for some unitary matrix U. It is the set of zeroes in $P_1(M_n(\mathbb{C}))$ of the associated Hermitian matrix

$$\mathscr{H}_e(r) = \begin{pmatrix} I & -P_0^* \\ -P_0 & P_0 P_0^* - r^2 I \end{pmatrix}. \tag{1.1}$$

It is a circle with radius r, centered at $\mathbf{P}_0 = (P_0 \;\; I)$.

THEOREM 1.1. *The center and the radius of a Euclidean circle are uniquely determined, and the associated Hermitian matrix is determined up to a nonzero scalar multiple.*

The proof will follow from the two lemmas:

LEMMA 1.2. *Let $\mathscr{H} \neq 0$ be a $2n \times 2n$ Hermitian matrix, so that $(P \;\; I)$ is a zero of $\mathscr{H}$, whenever it is a zero of $\tilde{\mathscr{H}}_e(r)$. Then there exists a nonzero real scalar a, so that $\mathscr{H} = a\tilde{\mathscr{H}}_e(r)$.*

Proof. Let $\mathscr{M}$ be the matrix $\left(\begin{smallmatrix} I & 0 \\ P_0 & I \end{smallmatrix}\right)$ that corresponds to the Möbius transformation $(Q \;\; I)\mathscr{M} = (Q + P_0 \;\; I)$. Let $\hat{\mathscr{H}} = \mathscr{M}\mathscr{H}\mathscr{M}^*$, then $\hat{\mathscr{H}}$ is a $2n \times 2n$ Hermitian matrix, and $(rU \;\; I)$ is a zero of $\hat{\mathscr{H}}$ for every unitary matrix U. In particular, $(\varepsilon rI \;\; I)$ are zeroes of $\hat{\mathscr{H}}$ for $\varepsilon = \pm 1, i$. Consequently, one verifies that if $\hat{\mathscr{H}} = \left(\begin{smallmatrix} \hat{H}_1 & \hat{H}_2 \\ \hat{H}_3 & \hat{H}_4 \end{smallmatrix}\right)$, where $\hat{H}_1$, $\hat{H}_2$, $\hat{H}_3$, $\hat{H}_4$ are $n \times n$ matrices, then

$$\hat{H}_2 = \hat{H}_3 = 0 \text{ and } r^2 U\hat{H}_1 U^* + \hat{H}_4 = 0 \text{ for every unitary matrix } U. \quad (1.2)$$

This last equality forces $\hat{H}_4$ to be a diagonal matrix (e.g., choose U to diagonalize $\hat{H}_1$). Consequently, for $U = I$, we get that $\hat{H}_1$ is a diagonal matrix. Using permutation matrices for U, we may conclude that both $\hat{H}_1$ and $\hat{H}_4$ are scalar matrices:

$$\hat{H}_1 = aI, \qquad \hat{H}_4 = bI, \qquad ar^2 + b = 0, \qquad \text{and} \qquad ab \neq 0. \qquad (1.3)$$

It results that $\mathscr{H} = \mathscr{M}^{-1}\hat{\mathscr{H}}\mathscr{M}^{-1*} = a\hat{\mathscr{H}}_e(r)$ as stated. ∎

LEMMA 1.3. *The center and the radius of a Euclidean circle are uniquely determined.*

Proof. Let $\gamma_e(\mathbf{P}_0, r) = \gamma_e(\mathbf{Q}_0, s)$. Then for $\varepsilon = \pm 1, i$ the points $(Q_0 + \varepsilon sI \;\; I)$ are points of $\gamma_e(P_0, r)$, that is,

$$(Q_0 + \varepsilon sI - P_0)(Q_0 + \varepsilon sI - P_0)^* = r^2 I, \qquad (1.4)$$

or

$$(Q_0 - P_0)(Q_0 - P_0)^* + \varepsilon s(Q_0 - P_0)^* + \bar{\varepsilon} s(Q_0 - P_0) + s^2 I = r^2 I, \quad (1.4')$$

whence it follows that $Q_0 = P_0$, and $s = r$ as stated. ∎

This completes the proof of the theorem. ∎

b. *The Spherical Circles*

We denote by $\hat{\mathbf{M}}$ the antipodal point to $\mathbf{M} = (M_1\ \ M_2)$, and we define the circle $\gamma_s(\hat{\mathbf{M}}, r)$ as the set of points $\mathbf{P}$ in $P_1(M_n(\mathbb{C}))$, for which $\mathbf{PM}^* = rU$, for some unitary matrix U. All matrices $\mathbf{Q}$ in this section are assumed to lie on K, that is $\mathbf{QQ}^* = I$ and every point in $P_1(M_n(\mathbb{C}))$ has representatives in K.

If $\mathbf{P}$ is on $\gamma_s(\hat{\mathbf{M}}, r)$, then $\mathbf{PM}^* = rV$, and $\mathbf{P}\hat{\mathbf{M}}^* = (1-r^2)^{1/2} W$, for some unitary matrices V, W.

It is the set of zeroes of the associated Hermitian matrix

$$\tilde{\mathscr{H}}_s(r) = \begin{pmatrix} M_1^* M_1 - r^2 I & M_1^* M_2 \\ M_2^* M_1 & M_2^* M_2 - r^2 I \end{pmatrix}. \tag{1.5}$$

This last remark is a consequence of

$$\tilde{\mathscr{H}}_s(r) = \mathscr{M}^* \mathscr{Q}_r \mathscr{M}, \qquad \text{where} \quad \mathscr{M} = \begin{pmatrix} \mathbf{M} \\ \hat{\mathbf{M}} \end{pmatrix}, \mathscr{Q}_r = \begin{pmatrix} (1-r^2) I & 0 \\ 0 & -r^2 I \end{pmatrix}. \tag{1.6}$$

Let $\mathscr{P} = \binom{\mathbf{P}}{\hat{\mathbf{P}}}$. If $\mathbf{PM}^* = rU_1$ then $\mathscr{P}\mathscr{M}^*$ is of the form

$$\begin{pmatrix} rU_1 & (1-r^2)^{1/2} U_2 \\ (1-r^2)^{1/2} U_3 & rU_4 \end{pmatrix}, \quad \text{where } U_1, U_2, U_3, U_4 \text{ are unitary matrices.} \tag{1.7}$$

We call such a $2n \times 2n$ unitary matrix *blockwise r-unitary matrix*.

Summarizing, $\mathbf{P}$ is a point on the circle $\gamma_s(\hat{\mathbf{M}}, r)$, of radius r, $(r = \chi(\mathbf{P}, \hat{\mathbf{M}}))$, centered at $\hat{\mathbf{M}}$ iff $\mathbf{P}$ annihilates $\mathscr{H}_s(r)$, and this is so iff $\mathscr{P}\mathscr{M}^*$ is blockwise r-unitary.

THEOREM 1.4. *The center and the radius of a spherical circle are determined up to antipodal equivalence, and the associated Hermitian matrix is determined up to a nonzero scalar multiple.*

By antipodal equivalence we understand that the pairs $(\hat{M}, r)$ and $(\hat{\mathbf{N}}, t)$ are related through $\hat{\mathbf{M}} = \hat{\mathbf{N}}$ and $r = t$, or else $\hat{\mathbf{M}} = \mathbf{N}$ and $r^2 + t^2 = 1$.

The proof of the theorem will follow from the two lemmas:

LEMMA 1.5. *Let $\mathscr{H} \neq 0$ be a $2n \times 2n$ Hermitian matrix so that $\mathbf{P}$ annihilates $\mathscr{H}$, whenever it is a zero of $\tilde{\mathscr{H}}_s(r)$. Then there exists a nonzero real scalar a, so that $\mathscr{H} = a\tilde{\mathscr{H}}_s(r)$.*

Proof. One may consider the forms $\mathscr{Q}_r$ and $\mathscr{M}\mathscr{H}\mathscr{M}^*$, and under the hypothesis, if $\mathbf{Q}$ is a zero of $\mathscr{Q}_r$ then it is a zero of $\mathscr{M}\mathscr{H}\mathscr{M}^*$. The points $(\varepsilon r I\ \ (1-r^2)^{1/2} I)$, for $\varepsilon = \pm 1, i$, are zeroes for $\mathscr{Q}_r$, whence of $\mathscr{M}\mathscr{H}\mathscr{M}^* = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix}$. It follows that

$$r^2 H_1 + r(1-r^2)^{1/2}(\bar{\varepsilon} H_3 + \varepsilon H_2) + (1-r^2) H_4 = 0 \qquad \text{for} \quad \varepsilon = \pm 1, i. \tag{1.8}$$

Consequently, $H_2 = H_3 = 0$, and there results for $(rU\ (1-r^2)^{1/2}I)$:

$$r^2 U H_1 U^* + (1-r^2) H_4 = 0 \qquad \text{for every unitary matrix } U. \tag{1.9}$$

As in the conclusion of Lemma 1.2 (from equality (1.2) on) it follows that H_1 and H_4 are scalar matrices:

$$H_1 = cI, \qquad H_4 = bI, \qquad cr^2 + b(1-r^2) = 0, \qquad \text{where} \quad cb \neq 0. \tag{1.10}$$

Therefore, $\mathscr{H} = a\mathscr{H}_s(r)$, where $a = c/(1-r^2) = -b/r^2$. ∎

Lemma 1.6. *Let $\gamma_s(\hat{\mathbf{M}}, r) = \gamma_s(\hat{\mathbf{N}}, t)$, then either $\hat{\mathbf{M}} = \hat{\mathbf{N}}$ and $r = t$, or else $\hat{\mathbf{M}} = \mathbf{N}$ and $r^2 + t^2 = 1$.*

Proof. Let $\mathscr{N} = \binom{\mathbf{N}}{\hat{\mathbf{N}}}$ and consider the associated Hermitian matrices $\mathscr{H}_s(r) = \mathscr{M}^* \mathscr{Q}_r \mathscr{M}$ and $\mathscr{H}_s(t) = \mathscr{N}^* \mathscr{Q}_t \mathscr{N}$. By Lemma 1.5,

$$\mathscr{N}^* \mathscr{Q}_t \mathscr{N} = a \mathscr{M}^* \mathscr{Q}_r \mathscr{M}, \tag{1.11}$$

or

$$\mathscr{W} \mathscr{Q}_t = a \mathscr{Q}_r \mathscr{W} \qquad \text{for} \quad \mathscr{W} = \mathscr{M}\mathscr{N}^* = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix}. \tag{1.11$'$}$$

We derive the following:

$$\begin{aligned} (1-t^2) W_1 &= a(1-r^2) W_1, & -t^2 W_2 &= a(1-r^2) W_2, \\ (1-t^2) W_3 &= -ar^2 W_3, & -t^2 W_4 &= -ar^2 W_4. \end{aligned} \tag{1.11$''$}$$

If $W_4 \neq 0$, then $W_1 \neq 0$ and we have the equalities

$$1 - t^2 = a(1-r^2) \qquad \text{and} \qquad -t^2 = -ar^2, \tag{1.12}$$

whence $a = 1$ and $t = r$. Furthermore, $W_2 = W_3 = 0$. Therefore $\hat{\mathbf{M}} = \hat{\mathbf{N}}$ in $P_1(M_n(\mathbb{C}))$.

If $W_4 = 0$, then $W_1 = 0$ and we get the equalities

$$1 - t^2 = -ar^2, \qquad -t^2 = a(1-r^2) \tag{1.12$'$}$$

whence $a = -1$, and $r^2 + t^2 = 1$. Furthermore, $\hat{\mathbf{M}} = \mathbf{N}$ in $P_1(M_n(\mathbb{C}))$. ∎

This completes the proof of the theorem. ∎

Corollary 1.7. *Let $\mathbf{P}$ lie on $\gamma_s(\hat{\mathbf{M}}, r)$, then $\hat{\mathbf{P}}$ lies on $\gamma_s(\hat{\mathbf{M}}, (1-r^2)^{1/2})$. In particular, if $\mathbf{P}$ and $\hat{\mathbf{P}}$ lie on $\gamma_s(\hat{\mathbf{M}}, r)$, then $r^2 = 1/2$.*

c. *The Non-Euclidean Circles*

The point $\mathbf{P}$ lies inside the unit disk Δ^-, if a point $\check{\mathbf{P}}$ exists (necessarily outside the unit disk, in Δ^+) so that $\mathscr{P}\mathscr{J}\mathscr{P}^* = \mathscr{J}$ where $\mathscr{P} = \binom{\check{\mathbf{P}}}{\mathbf{P}}$ and $\mathscr{J} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. We call $\mathscr{P}$ the $\mathscr{J}$-unitary completion of $\mathbf{P}$. Here we identified the points in Δ^- and Δ^+ with their corresponding representatives in $\mathscr{J}^-$ and $\mathscr{J}^+$.

The circle $\gamma_n(\mathbf{M}, r)$-centered at $\mathbf{M} = (M_3 M_4) \in \Delta^-$, and of radius r, is the set of points $\mathbf{P}$ for which $\mathbf{P}\mathscr{J}\mathbf{M}^* = (1+\rho^2)^{1/2} U$ and $\mathbf{P}\mathscr{J}\check{\mathbf{M}}^* = \rho V$, where U, V are unitary matrices, and $\rho = sinhr$. We may replace the condition $\mathbf{P}\mathscr{J}\check{\mathbf{M}}^* = \rho V$ with the equivalent condition $\mathbf{P} \in \Delta^-$. The pseudochordal distance, corresponding to the distance r, is $\rho/(1+\rho^2)^{1/2}$.

The associated Hermitian matrix is

$$\tilde{\mathscr{H}}_n(r) = \begin{pmatrix} (1+\rho^2)I + M_3^* M_3 & -M_3^* M_4 \\ -M_4^* M_3 & -(1+\rho^2)I + M_4^* M_4 \end{pmatrix}. \tag{1.13}$$

A $\mathscr{J}$-unitary matrix $\mathscr{U}$ is *blockwise r-$\mathscr{J}$-unitary* if

$$\mathscr{U} = \begin{pmatrix} (1+r^2)^{1/2} U_1 & rU_2 \\ rU_3 & (1+r^2)^{1/2} U_4 \end{pmatrix},$$

$$\text{where } U_1,\ U_2,\ U_3,\ U_4 \text{ are unitary matrices,} \tag{1.14}$$

and the matrix $\mathscr{U}$ satisfies $\mathscr{U}\mathscr{J}\mathscr{U}^* = \mathscr{J}$.

The point $\mathbf{P}$ is on $\gamma_n(\mathbf{M}, r)$ iff $\mathscr{P}\mathscr{J}\mathscr{M}^*$ is blockwise r-$\mathscr{J}$-unitary.

We have $\tilde{\mathscr{H}}_n(r) = \mathscr{M}^* \mathscr{Q}_\rho \mathscr{M}$, where

$$\mathscr{Q}_\rho = \begin{pmatrix} (1+\rho^2)I & 0 \\ 0 & -\rho^2 I \end{pmatrix}. \tag{1.15}$$

THEOREM 1.8. *The center and the radius of a non-Euclidean circle are uniquely determined. The associated Hermitian matrix is determined up to a nonzero scalar multiple.*

The proof of the theorem will follow from the two lemmas:

LEMMA 1.9. Let $\mathscr{H} \neq 0$ *be a* $2n \times 2n$ *Hermitian matrix so that* $\mathbf{P}$, $\mathbf{P} \in \Delta^-$, *is a zero of* $\mathscr{H}$ *whenever it is a zero of* $\tilde{\mathscr{H}}_n(r)$. *Then there exists a nonzero real scalar a, so that* $\mathscr{H} = a\mathscr{H}_n^*(r)$.

Proof. Under the hypothesis $\mathbf{Q}$, $\mathbf{Q} \in \Delta^-$, is a zero of $\mathscr{J}\mathscr{M}^*\mathscr{H}\mathscr{M}\mathscr{J}$ whenever it is a zero of $\mathscr{Q}_\rho$, where $\rho = \sinh r$. Let

$$\mathscr{J}\mathscr{M}^*\mathscr{H}\mathscr{M}\mathscr{J} = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix}. \tag{1.16}$$

Note that $\mathbf{Q} = (\rho U \ (1+\rho^2)^{1/2}\,\varepsilon I)$ is a zero of $\mathscr{Q}_\rho$ for every unitary matrix U, and $\varepsilon = \pm 1, i$, and $\mathbf{Q} \in \Delta^-$. It therefore follows that

$$\rho^2 U H_1 U^* + \rho(1+\rho^2)^{1/2}(\bar{\varepsilon} U H_2 + \varepsilon H_3 U^*) + (1+\rho^2) H_4 = 0. \quad (1.17)$$

Consequently, $H_2 = H_3 = 0$. From $\rho^2 U H_1 U^* + (1+\rho^2)\, H_4 = 0$, for every unitary matrix U, it follows, as in the conclusion of Lemma 1.2 (from equality (1.2) on) that both H_1 and H_4 are scalar matrices,

$$H_1 = cI, \qquad H_4 = bI. \qquad \rho^2 c + (1+\rho^2) b = 0, \qquad \text{and} \qquad bc \neq 0. \qquad (1.18)$$

Thus $\mathscr{H} = a\mathscr{H}_n(r)$, where $a = c/(1+\rho^2) = -b/\rho^2$. ∎

LEMMA 1.10. *Let* $\gamma_n(\mathbf{M}, r) = \gamma_n(\mathbf{N}, t)$, *then* $\mathbf{M} = \mathbf{N}$ *and* $r = t$.

Proof. Let $\mathscr{N} = \binom{\tilde{\mathbf{N}}}{\mathbf{N}}$, and consider the associated Hermitian matrix $\check{\mathscr{H}}_n(t) = \mathscr{N}\mathscr{Q}_\tau\mathscr{N}^*$ ($\tau = \sinh t$). By Lemma 1.9, $\check{\mathscr{H}}_n(t) = a\tilde{\mathscr{H}}_n(r)$, whence the equality:

$$\mathscr{W}\mathscr{Q}_\tau = a\mathscr{Q}_\rho\mathscr{W}, \qquad \text{where} \quad \mathscr{W} = \mathscr{M}^*\mathscr{N} = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix}. \qquad (1.19)$$

There result the equalities:

$$\begin{aligned} (1+\tau^2) W_1 &= a(1+\rho^2) W_1, & -\tau^2 W_2 &= a(1+\rho^2) W_2, \\ (1+\tau^2) W_3 &= -\rho^2 W_3, & -\tau^2 W_4 &= -\rho^2 W_4. \end{aligned} \qquad (1.20)$$

$\mathscr{W}$ is a $\mathscr{J}$-unitary matrix, therefore if $W_1 = 0$ then $W_4 = 0$ and this leads to $a = 1$ and $1 + \rho^2 + \tau^2 = 0$—a contradiction.

Thus $W_1 \neq 0$, and also $W_4 \neq 0$. This yields the equalities: $a = 1$ and $\rho = \tau$. Furthermore, $W_2 = W_3 = 0$. Consequently, W_1 and W_4 are unitary matrices. Thus $\mathbf{M} = \mathbf{N} \in \Delta^- \subset P_1(M_n(\mathbb{C}))$. ∎

This completes the proof of the theorem. ∎

2. APPLICATIONS

a. *Special Möbius transformations of the unit disk* Δ^-

Let A, B, C, D be $n \times n$ matrices, and let

$$\mathscr{M} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

be the matrix corresponding to the Möbius transformation $\mathscr{M}$. The image of the point $\mathbf{P}$ will be written as $\mathcal{M}(\mathbf{P})$, or $\mathbf{P}\mathcal{M}$, where the last notation is the standard matrix multiplication.

We let $\mathbf{P} \in \Delta^-$, and we set $\mathbf{P} = (P_3 \ P_4)$. Hence $P_4 P_4^* - P_3 P_3^* > 0$.

We use the convention of denoting the Möbius transformation by $\mathcal{M}$, as well as the corresponding matrix. Recall that the matrix is determined up to a (nonzero) scalar factor. In particular, the scalar is properly chosen when we assume that $\mathcal{M}\mathcal{J}\mathcal{M}^* = \mathcal{J}$.

THEOREM 2.1. *Let $\mathcal{M}$ be a Möbius transformation that carries the unit disk Δ^- into itself. The following conditions are equivalent*:

(i) *$\mathcal{M}$ maps Δ^- onto Δ^-.*

(ii) *$\mathcal{M}$ maps Δ^+ into Δ^+.*

(iii) *$\mathcal{M}$ preserves the non-Euclidean distance in Δ^-.*

(iv) *$\mathcal{M}$ satisfies $\mathcal{M}\mathcal{J}\mathcal{M}^* = \mathcal{J}$.*

To prove this theorem we need some more insight into the transformations that carry Δ^- into Δ^-.

Let $\mathcal{M}$, $\mathcal{N}$ be Möbius transformations. We say that *$\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{J}$-equivalent, if there exist $\mathcal{J}$-unitary matrices $\mathcal{H}$ and $\mathcal{K}$, so that $\mathcal{H}\mathcal{M}\mathcal{K} = \mathcal{N}$.* As the $\mathcal{J}$-unitary matrices form a group, we defined indeed an equivalence relation.

We say that *$\mathcal{M}$ is a diagonal map, iff $\mathcal{M}$ is a diagonal matrix, $\mathcal{M} = \left(\begin{smallmatrix} M_1 & 0 \\ 0 & M_2 \end{smallmatrix}\right)$, $M_i = \operatorname{diag}\{m_{i1}, \ldots, m_{in}\}$,* for $i = 1, 2$. *The diagonal map matrix $\mathcal{M}$ is a contraction if* $0 < m_{1i} \leqslant m_{2j}$, $\forall_{i,j}$ (in particular, m_{ij} are all real positive numbers).

Note that being a diagonal (diagonal contracting) map is independent of the choice of the matrix representative of the map (as long as it is real).

Since $\mathcal{J}$-unitary matrices carry $\Delta^-(\Delta^+)$ isomorphically onto $\Delta^-(\Delta^+)$, and keep the chordal distance invariant, it follows that if $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{J}$-equivalent, then $\mathcal{M}$ satisfies (i) ((ii), (iii), (iv)) of Theorem 2.1 iff $\mathcal{N}$ satisfies (i) ((ii), (iii), (iv)). Also, $\mathcal{M}$ carries Δ^- into Δ^- iff $\mathcal{N}$ does so.

LEMMA 2.2. *Let $\mathcal{M}$ be a Möbius transformation. Then $\mathcal{M}$ is $\mathcal{J}$-equivalent to a diagonal contracting map iff $\mathcal{M}$ maps Δ^- into Δ^-.*

Proof. Suppose that $\mathcal{M} = \left(\begin{smallmatrix} M_1 & 0 \\ 0 & M_2 \end{smallmatrix}\right)$ is a diagonal contracting map, and let $(P \ Q) \in \mathcal{J}^-$. Its image under $\mathcal{M}$ is $(PM_1 \ QM_2)$. Given any vector v, $v \in \mathbb{C}^n$. Let $u = vP$, $w = vQ$. Since $-PP^* + QQ^* = I$, it follows that $ww^* - uu^* > 0$. Let $uM_1 = \bar{u}$, $wM_2 = \bar{w}$. Then

$$\bar{u}\bar{u}^* \leqslant k^2 uu^*, \qquad \text{where} \quad k = \max_i m_{1i},$$

and

$$\bar{w}\bar{w}^* \geqslant l^2 ww^*, \qquad \text{where} \quad l = \min_i m_{2i}$$

Since M is a diagonal contracting map, $0 < k \leqslant l$, then $\bar{w}\bar{w}^* - \bar{u}\bar{u}^* > l^2 ww^* - k^2 uu^* > l^2(ww^* - uu^*) > 0$.

Hence $-(PM_1)(PM_1)^* + (QM_2)(QM_2)^* > 0$, and thus, $(PM_1 \; QM_2) \in \Delta^-$ and the transformation maps Δ^- into Δ^-.

Conversely, let $\mathscr{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a Möbius transformation that carries Δ^- into Δ^-.

In particular $(0 \; I)\mathscr{M} = (C \; D) \in \Delta^-$, or $C = PC_2$, $D = PD_2$, $(C_2 \; D_2) \in \mathscr{J}^-$, and $|P| \neq 0$. Let $(C_1 \; D_1) = (C_2 \; D_2)$, then $\mathscr{N} = \begin{pmatrix} C_1 & D_1 \\ C_2 & D_2 \end{pmatrix}$ is a $\mathscr{J}$-unitary matrix, and $\mathscr{M}\mathscr{N}^{-1} = \begin{pmatrix} A_0 & B_0 \\ 0 & D_0 \end{pmatrix}$ represents a Möbius transformation that carries Δ^- into Δ^-.

Given any scalar λ, $\mathbf{Q} = (\lambda A_0^{-1} \; Q) \in \mathscr{J}^-$ if we choose Q to satisfy $QQ^* - |\lambda|^2 A_0^{-1} A_0^{-1*} = I$. In particular, $|Q| \neq 0$. The image of $\mathbf{Q}$ under $\mathscr{M}\mathscr{N}^{-1}$, is $(\lambda I \; \lambda A_0^{-1}B_0 + QD_0)$, and this point lies in Δ^- iff:

$$f(\lambda) = (\lambda A_0^{-1}B_0 + QD_0)(\lambda A_0^{-1}B_0 + QD_0)^* - |\lambda|^2 I > 0. \tag{2.1}$$

If $|B_0| \neq 0$, there exists a scalar ν, $\nu \neq 0$, such that $|D_0^{-1}Q^{-1}A_0^{-1}B_0 + \nu I| = 0$, whence $|A_0^{-1}B_0 + \nu QD_0| = 0$. If we choose $\lambda_0 = \nu^{-1}$, then $|\lambda_0 A_0^{-1}B_0 + QD_0| = 0$, and $f(\lambda_0)$ is not positive which contradicts (2.1). If $B_0 \neq 0$ and $|B_0| = 0$, then the contradiction follows by continuity. Hence $B_0 = 0$, and it follows that $\mathscr{M}$ is $\mathscr{J}$-equivalent to $\begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix}$. Using $\mathscr{J}$-unitary matrices of the form $\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$, where U, V are unitary matrices, it follows that $\mathscr{M}$ is $\mathscr{J}$-equivalent to a real (positive) diagonal matrix $\mathscr{N} = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$.

Let μ be any real number, let E_i be the matrix whose jk-entry is $(E_i)_{jk} = \delta_{ij}\,\delta_{ik}$. Let $Q_i(\mu)$ be the matrix whose jk-entry is $(Q_i(\mu))_{jk} = \delta_{jk}(1 + \delta_{ij}((1+\mu^2)^{1/2} - 1))$.

Let $\mathbf{Q}_i(\mu) = (\mu E_i Q_i(\mu))$. Since $E_i^2 = E_i$, and $Q_i(\mu) = I + ((1+\mu^2)^{1/2} - 1)E_i$, one verifies by straightforward computation that $\mathbf{Q}_i(\mu) \in \mathscr{J}^-$.

By assumption, the image of $\mathbf{Q}_i(\mu)$ under $\mathscr{N}$ is in Δ^-, that is, $(\mu E_i N_1 \; Q_i(\mu) N_2) \in \Delta^-$, that is,

$$-|\mu n_{1i}|^2 + |(1+\mu^2)^{1/2} n_{2i}|^2 > 0 \tag{2.2}$$

and for this inequality to hold for all μ's, it follows that $n_{1i} \leqslant n_{2i}$ for $i = 1,\ldots, n$.

Finally, using J-unitary matrices of the form $\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$, where U, V are permutation matrices, we may derive

$$n_{1i} \leqslant n_{2j} \qquad \text{for} \quad i, j = 1,\ldots, n. \tag{2.2$'$}$$

and this completes the proof of the lemma. ▌

An immediate consequence of the last part of the above proof is:

COROLLARY 2.3. *If $\mathcal{M}$ carries Δ^- into Δ^-, then any real positive diagonal matrix $\mathcal{T}$ that is $\mathcal{J}$-equivalent to $\mathcal{M}$ is a contraction.*

LEMMA 2.4. *Let $\mathcal{M}$ map Δ^- (into, and) onto Δ^-. Then $\mathcal{M}$ is $\mathcal{J}$-equivalent to the identity.*

Proof. By Lemma 2.2 $\mathcal{M}$ is equivalent to a contraction $\mathcal{N} = \left(\begin{smallmatrix} N_1 & 0 \\ 0 & N_2 \end{smallmatrix}\right)$. Since $\mathcal{M}$ maps Δ^- onto Δ^-, so does $\mathcal{N}$, and hence $\mathcal{N}^{-1}$ maps Δ^- into Δ^-. Thus $\left(\begin{smallmatrix} N_1^{-1} & 0 \\ 0 & N_2^{-1} \end{smallmatrix}\right)$ is a contraction, in particular the corollary implies $n_{1i}^{-1} \leqslant n_{2j}^{-1}$, whence $v = n_{1i} = n_{2i}$ for $i = 1,..., n$, whence $N = vI$. Hence $\mathcal{N}$ may be represented by the identity matrix.

LEMMA 2.5. *Let $\mathcal{M}$ map Δ^- into Δ^-, and Δ^+ into Δ^+. Then $\mathcal{M}$ is $\mathcal{J}$-equivalent to the identity.*

Proof. By Lemma 2.2, $\mathcal{M}$ is $\mathcal{J}$-equivalent to a contraction $\mathcal{N} = \left(\begin{smallmatrix} N_1 & 0 \\ 0 & N_2 \end{smallmatrix}\right)$. It is straightforward, that $\mathcal{N}$ maps Δ^+ into Δ^+ iff $\mathcal{N}' = \left(\begin{smallmatrix} N_2 & 0 \\ 0 & N_1 \end{smallmatrix}\right)$ maps Δ^- into Δ^-. Applying Lemma 2.2 to $\mathcal{N}'$ it follows that $n_{2i} \leqslant n_{1j}$ for $i, j = 1,..., n$, whence $\mathcal{N}$ is a scalar matrix. It follows that $\mathcal{M}$ is $\mathcal{J}$-equivalent to the identity matrix.

LEMMA 2.6. *Let $\mathcal{M}$ map Δ^- into Δ^-, and assume that $\mathcal{M}$ preserves the chordal distance. Then $\mathcal{M}$ is $\mathcal{J}$-equivalent to the identity matrix.*

Proof. By Lemma 2.2, $\mathcal{M}$ is $\mathcal{J}$-equivalent to a contraction $\mathcal{N} = \left(\begin{smallmatrix} N_1 & 0 \\ 0 & N_2 \end{smallmatrix}\right)$. We consider the point $\mathbf{Q}_i(\mu)$, as in the proof of Lemma 2.2. Its distance from $(0\ I)$ is μ. (The pseudo-chordal distance is actually given by $\mu/(1+\mu_2)^{1/2}$. The distance between their images under $\mathcal{N}$ is μ,

$$\mu n_{1i}/((1+\mu^2)\, n_{2i}^2 - \mu^2 n_{1i}^2)^{1/2} = \mu, \tag{2.3}$$

that is,

$$n_{1i}^2 = (1+\mu^2)\, n_{2i}^2 - \mu^2 n_{1i}^2 \qquad \text{or} \qquad n_{1i} = n_{2i}. \tag{2.3$'$}$$

and it follows from $n_{1i} \leqslant n_{2j}$ that $v = n_{1i} = n_{2j}$ for $i, j = 1,..., n$. Hence $\mathcal{N}$ is a scalar matrix, and $\mathcal{M}$ is $\mathcal{J}$-equivalent to the identity matrix.

Returning to the proof of Theorem 2.1, and since (iv) $\to$ (i), (ii), and (iii) (cf. [3]), the validity of the theorem is a consequence of Lemmas 2.4, 2.5, and 2.6. ▮

Since the $\mathcal{J}$-unitary matrices form a group, it follows that under the hypothesis of Theorem 2.1, the following holds:

LEMMA 2.7. *The Möbius transformation $\mathcal{M}$ can be represented by a $\mathcal{J}$-unitary matrix iff $\mathcal{M}$ is $\mathcal{J}$-equivalent to a $\mathcal{J}$-unitary matrix.*

b. *Special Transformations in* $P_1(M_n(\mathbb{C}))$

The finite points which annihilate $\mathscr{J}$, are the unitary matrices. Thus we refer to the zeros of $\mathscr{J}$ in $P_1(M_n(\mathbb{C}))$ as unitary points:

$$(P_1 \;\; P_2) \qquad \text{is } \textit{unitary} \text{ if} \quad P_1 P_1^* = P_2 P_2^*.$$

The set *of unitary points are* referred to as *the unit circle.*

The finite points which annihilate the form $\mathscr{H} = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}$ are the Hermitian matrices. We refer to the zeros of $\mathscr{H}$ in $P_1(M_n(\mathbb{C}))$ as Hermitian points: $(P_1 \;\; P_2)$ is *Hermitian* if $P_1 P_2^* = P_2 P_1^*$. *The set of Hermitian points is* referred to as *the Hermitian line.*

Let $\mathscr{K} = 2^{-1/2}\begin{pmatrix} iI & I \\ I & iI \end{pmatrix}$, then we have the equalities.

$$\mathscr{K}\mathscr{H}\mathscr{K}^* = \mathscr{J}, \tag{2.4}$$

$$\mathscr{K}\mathscr{J}\mathscr{K}^* = -\mathscr{H}, \tag{2.5}$$

$$\mathscr{K}^*\mathscr{J}\mathscr{K} = \mathscr{H}, \tag{2.6}$$

$$\mathscr{K}\mathscr{J}\mathscr{K} = -\mathscr{J}. \tag{2.7}$$

Remark that $\mathscr{K}$ is a unitary matrix.

We will characterize special Möbius maps on $P_1(M_n(\mathbb{C}))$. One can easily obtain analoguous results for maps on Δ^- or on the Euclidean plane.

PROPOSITION 2.8. *Let $\mathscr{M}$ map the unit circle onto itself, so that Δ^- maps into Δ^-, then $\mathscr{M}$ can be represented by a $\mathscr{J}$-unitary matrix. If $\mathscr{M}$ maps Δ^- into Δ^+, then $\mathscr{M}\mathscr{J}\mathscr{M}^* = -\mathscr{J}$ (for a suitable choice of $\mathscr{M}$).*

Proof. By the hypothesis, the zeros of $\mathscr{M}\mathscr{J}\mathscr{M}^*$ and $\mathscr{J}$ are the same whence $\mathscr{M}\mathscr{J}\mathscr{M}^* = t\mathscr{J}$ for some scalar t, $t \neq 0$. Upon normalizing $\mathscr{M}$, we may assume that t is real, and $t^2 = 1$. The result now follows. ∎

Remark that if $\mathscr{M} = \begin{pmatrix} \check{\mathbf{P}} \\ \mathbf{P} \end{pmatrix}$, then the map uniquely determines $\mathbf{P}$, but different representatives for $\mathbf{P}$ yield in general different maps.

PROPOSITION 2.9. *Let $\mathscr{M}$ map the Hermitian line onto itself, so that the upper half plane maps into itself. Then $\mathscr{N} = \mathscr{K}\mathscr{M}\mathscr{K}^*$ is $\mathscr{J}$-unitary. If the upper half-plane maps into the lower half-plane, then $\mathscr{N}\mathscr{J}\mathscr{N}^* = -\mathscr{J}$.*

Proof. By the hypothesis, $\mathscr{M}\mathscr{H}\mathscr{M}^*$ and $\mathscr{H}$ have the same set of zeros, so that $\mathscr{M}\mathscr{H}\mathscr{M}^* = t\mathscr{H}$ for some scalar t. It follows, using 2.6, that $(\mathscr{K}\mathscr{M}\mathscr{K}^*)\,\mathscr{J}(\mathscr{K}\mathscr{M}\mathscr{K}^*) = t\mathscr{J}$, hence as in Proposition 2.8 we may normalize $\mathscr{M}$ so that t is real and $t^2 = 1$. The conclusion now follows. ∎

Note that $\mathscr{M} = \mathscr{K}^*\mathscr{N}\mathscr{K}$ with $\mathscr{N}\mathscr{J}\mathscr{N}^* = \pm\mathscr{J}$.

PROPOSITION 2.10. *Let $\mathcal{M}$ map the unit circle into the Hermitian line, then $\mathcal{N} = \mathcal{M}\mathcal{K}^*$ is $\mathcal{J}$-unitary, if the inside of the unit disk maps into the upper half-plane. If it maps into the lower half-plane, then $\mathcal{N}\mathcal{J}\mathcal{N}^* = -\mathcal{J}$.*

Proof. The form $\mathcal{M}\mathcal{H}\mathcal{M}^*$ has for its zeros the unit circle, whence for some scalar t we have the equality $\mathcal{M}\mathcal{H}\mathcal{M}^* = t\mathcal{J}$. Upon normalization, we may derive that t is real, and $t^2 = 1$. The case $t = 1$ corresponds to the case that the inside of the unit disk maps into the upper half-plane, and the case $t = -1$ to that when the map is into the lower half-plane. By 2.6, we may write $(\mathcal{M}\mathcal{K}^*)\,\mathcal{J}(\mathcal{K}\mathcal{M}^*) = t\mathcal{J}$, and this completes the proof. ▮

Note that we have $\mathcal{M} = \mathcal{N}\mathcal{K}$, with $\mathcal{N}\mathcal{J}\mathcal{N}^* = \pm\mathcal{J}$.

PROPOSITION 2.11. *Let $\mathcal{M}$ map the Hermitian line into the unit circle. Then $\mathcal{N} = \mathcal{K}\mathcal{M}$ is $\mathcal{J}$-unitary, if the upper half-plane maps into the unit disk. If it maps into the outside of the unit disk, then $\mathcal{N}\mathcal{J}\mathcal{N}^* = -\mathcal{J}$.*

Proof. The form $\mathcal{M}\mathcal{J}\mathcal{M}^*$ is annihilated by the set of zeros of $\mathcal{H}$, whence for some scalar t, $\mathcal{M}\mathcal{J}\mathcal{M}^* = t\mathcal{H}$. By 2.4, this implies $(\mathcal{K}\mathcal{M})$-$\mathcal{J}(KM)^* = t\mathcal{J}$. Upon normalization, we may derive that t is real, and $t^2 = 1$. The conclusion now follows in a similar way to the conclusion of Proposition 2.10.

Remark that the converses of Propositions 2.8, 2.9, 2.10, and 2.11 hold.

c. *Comparison of Euclidean, Spherical, and Non-Euclidean Circles*

We shall compare the circles in two ways: first, which of the Euclidean circles are spherical ones, and second, what are the relations between the centers and radii of this circle in the different geometries. We shall also discuss the relations with the non-Euclidean circles.

We consider the Euclidean circle given by (1.1), $\mathcal{H}_e(r_1)$, the spherical circle given by (1.5), $\mathcal{H}_s(r_2)$, and the non-Euclidean circle given by (1.13), $\mathcal{H}_n(r_3)$. The center of $\mathcal{H}_e(r_1)$ is $(P_0\ \ I)$ and its radius r_1. The center of $\mathcal{H}_s(r_2)$ is $\widehat{(M_1\ \ M_2)}$ and its radius is r_2, or, the center is $(M_1\ \ M_2)$ and the radius is $(1 - r_2^2)^{1/2}$. The center of $\mathcal{H}_n(r_3)$ is $(M_3\ \ M_4)$, and its radius is r_3. In the comparison, we will consider $\rho = \sinh r_3$ as the radius of $\mathcal{H}_n(r_3)$. To avoid confusion, we will replace M_3 by N_3 and M_4 by N_4. The matrix $\mathcal{M}$ will denote the unitary completion as in (1.6), and $\mathcal{N}$ will denote the $\mathcal{J}$-unitary completion of $\mathbf{N} = (N_3\ \ N_4)$ [cf. 3].

PROPOSITION 2.12. *Let $\mathcal{H}_e(r_1)$ and $\mathcal{H}_s(r_2)$ have the same set of zeros. Then P_0 is a scalar multiple of a unitary matrix, and $\mathcal{M}$ is a blockwise a-unitary matrix (cf. (1.7)). If $P_0 = bU$, $M_1 = aV$, and $M_2 = (1 - a^2)^{1/2} W$, with U, V, W unitary matrices, then the following relations hold: $U = W^*V$, $b = a(1 - a^2)^{1/2}/(r_2^2 - a^2)$, $r_1 = r_2(1 - r_2^2)^{1/2}/|r_2^2 - a^2|$ (under the hypothesis,* *it is impossible to have $r_2^2 = a^2$) $a^2 = [1 - t(b^2 - r_1^2 - 1)]/2$, $r_2^2 = [1 - t(b^2 - r_1^2 + 1)]/2$, where t satisfies $t^2[(b^2 - r_1^2 - 1)^2 + 4b^2] = 1$ (see Lemma* 1.6 *for the two possible choices for t).*

Proof. The result follows by comparing $t\mathscr{H}_e(r_1) = \mathscr{H}_s(r_2)$, noting that $tI = M_1^* M_1 - r_2^2 I$ implies that $M_1 = aV$, whence $M_2 = (1 - a^2)^{1/2} W$ for suitable unitary matrices V, W. Furthermore, $-tP_0 = M_2^* M_1$ implies $P_0 = bU$, where $U = W^* V$ once a, b are positive and t properly chosen.

The rest follows as the solution for the derived set of equalities:

$$
\begin{aligned}
t &= a^2 - r_2^2, \\
-tb &= a(1 - a^2)^{1/2}, \\
t(b^2 - r_1^2) &= 1 - a^2 - r_2^2. \quad \blacksquare
\end{aligned}
\tag{2.8}
$$

Note in particular, that if $P_0 = 0$, then $b = 0$. In this case $a = 0$ and $r_2 = (1 + r_1^2)^{-1/2}$, or else $a = 1$ and $r_2 = r_1/(1 + r_1^2)^{1/2}$.

PROPOSITION 2.13. *Let $\mathscr{H}_e(r_1)$ and $\mathscr{H}_n(r_3)$ have the same set of zeros. Then P_0 is a scalar multiple of a unitary matrix, and $\mathscr{N}$ is a blockwise c-$\mathscr{J}$-unitary matrix. If $P_0 = dU$, $\mathscr{N}_3 = cV$, and $\mathscr{N}_4 = (1 + c^2)^{1/2} W$, with U, V, W unitary matrices, then the following relations hold: $U = W^* V$, $d = c(1 + c^2)^{1/2}/(1 + \rho^2 + c^2)$, $r_1 = \rho(1 + \rho^2)^{1/2}/(1 + \rho^2 + c^2)$ and we have $d + r_1 < 1$. Furthermore,*

$$
c^2 = [t(d^2 - r_1^2 + 1) - 1]/2 \qquad \rho^2 = [t(r_1^2 - d^2 + 1) - 1]/2,
$$

where

$$
t^2[(1 - (d - r_1)^2)(1 - (d + r_1)^2)] = 1 \quad \text{and} \quad (d + r_1)^2 < 1, \quad (d - r_1)^2 < 1
$$

(c, d, r_1, ρ are all nonnegative).

Proof. The result follows by comparing $t\mathscr{H}_e(r_1) = \mathscr{H}_n(r_3)$, noting that $tI = (1 + \rho^2) I + N_3^* N_3$ implies that $N_3 = cV$, whence $N_4 = (1 + c^2)^{1/2} W$ for unitary matrices V, W. Furthermore, the equality $-tP_0 = M_4^* M_3$ implies $P_0 = dU$, with $U = W^* V$. The above comparison yields the following equalities:

$$
\begin{aligned}
t &= 1 + \rho^2 + c^2, \\
-td &= -c(1 + c^2)^{1/2}, \\
t(d^2 - r_1^2) &= -(1 + \rho^2) + (1 + c^2).
\end{aligned}
\tag{2.9}
$$

The conclusion now follows by solving for d and r_1 when ρ and c are given, and solving for ρ and c when d and r_1 are known. $\blacksquare$

Note that in particular, if $P_0=0$, then $d=0$. In this case $c=0$ and $\rho=r_1/(1-r_1^2)^{1/2}$.

PROPOSITION 2.14. *Let $\mathscr{H}_s(r_2)$ and $\mathscr{H}_n(r_3)$ have the same set of zeros. Then $\mathscr{M}$ is a blockwise a-unitary matrix, and $\mathscr{N}$ is a blockwise c-$\mathscr{J}$-unitary matrix. If $M_1=aW_1$, $M_2=(1-a^2)^{1/2}W_2$ and $N_3=cW_3$, $N_4=(1+c^2)^{1/2}W_4$, where W_1, W_2, W_3, W_4 are unitary matrices. The following relations hold:*

(i) $c^2=[t(1-2r_2^2)-1]/2$, $\rho^2=[t(2a^2-1)-1]/2$, *where t satisfies $t^2[(1-2r_2^2)^2-4a^2(1-a^2)]=1$, and the following restrictions hold: $(1-2r_2^2)<0$, $2a^2-1<0$. Furthermore, we choose the negative root for t.*

(ii) $r_2^2=[1-\tilde{t}(1+2c^2)]/2$, $a^2=[1+\tilde{t}(1+2\rho^2)]/2$, *where $\tilde{t}$ satisfies $\tilde{t}^2[4c^2(1+c^2)+(1+2\rho^2)^2]=1$. We choose for $\tilde{t}$ the negative root.*

Proof. Comparing $t\mathscr{H}_s(r_2)=\mathscr{H}_n(r_3)$, and using the equalities

$$M_1=U_1L_1V_1,\qquad M_2=U_1L_2V_2,\qquad L_1^2+L_2^2=I,$$
$$N_3=U_2K_2V_3,\qquad N_4=U_2K_1V_4,\qquad K_1^2-K_2^2=I,$$

where U_1, U_2, V_1, V_2, V_3, V_4 are unitary matrices, and L_1, L_2, K_1, K_2, are nonnegative diagonal matrices, the following equalities result:

$$t(V_1^*L_1^2V_1-r_2^2I)=(1+\rho^2)I+V_3^*K_2^2V_3, \tag{2.10}$$
$$tV_1^*L_1L_2V_2=-V_3^*K_2K_1V_4, \tag{2.11}$$
$$t(V_2^*L_2^2V_2-r_2^2I)=V_4^*K_1^2V_4-(1+\rho^2)I, \tag{2.12}$$

it follows that $V_1V_3^*K_2^2V_3V_1^*$, $V_2V_4^*K_1^2V_4V_2^*$ are diagonal matrices, and $tL_1L_2=-K_2K_1$ (by properly choosing the order of the elements in K_1 and K_2). Hence $t<0$. From (2.11), setting $tL_1L_2=D$, we have $V_3V_1^*D=DV_4V_2^*$. Note that $|K_1|\neq 0$, whence $V_3V_1^*K_2=K_2V_4V_2^*=V_4V_2^*K_2=K_2V_3V_1^*$.

We derive now from (2.10) and (2.12), using $K_1^2=I+K_2^2$, the equality:

$$\begin{aligned}t(I-2r^2I)&=I+V_1V_3^*K_2^2V_3V_1^*+V_2V_4^*K_2^2V_4V_2^*\\&=I+2V_1V_3^*K_2^2V_3V_1^*\end{aligned}$$

which readily implies that K_2 is a scalar matrix, $K_2=cI$, whence $K_1=(1+c^2)^{1/2}I$. Consequently L_1 and L_2 are scalar matrices $L_1=aI$ $L_2=(1-a^2)^{1/2}I$. We have thus: $M_1=aW_1$, $M_2=(1-a^2)^{1/2}W_2$, $N_3=cW_3$, $N_4=(1+c^2)^{1/2}W_4$ for unitary matrices W_1, W_2, W_3, W_4.

The equalities above become

$$t(a^2 - r_2^2) = 1 + \rho^2 + c^2, \tag{2.10$'$}$$

$$ta(1 - a^2)^{1/2} = -c(1 + c^2)^{1/2}, \tag{2.11$'$}$$

$$t(1 - a^2 - r_2^2) = (1 + c^2) - (1 + \rho^2). \tag{2.12$'$}$$

The condition $t < 0$ fixes the choice of the center and the radius, (if $t > 0$, interchange a^2 by $(1 - a^2)$ and r_2^2 by $(1 - r_2^2)$).

The equalities stated as (i) are the solutions of this system.

To get part (ii), we consider

$$a^2 - r_2^2 = \tilde{t}(1 + \rho^2 + c^2), \tag{2.10$''$}$$

$$a(1 - a^2)^{1/2} = -\tilde{t}c(1 + c^2)^{1/2}, \tag{2.11$''$}$$

$$1 - a^2 - r_2^2 = \tilde{t}[(1 + c^2) - (1 + \rho^2)], \tag{2.12$''$}$$

and we may choose $\tilde{t} < 0$. This completes the proof. ∎

Note that for $a = 0$, we have $c = 0$ and $\rho^2 = (1 + r_2^2)/(2r_2^2 - 1)$, $(2r_2^2 - 1 > 0)$. If $c = 0$, then $a = 0$ and $r_2^2 = (1 + \rho^2)/(1 + 2\rho^2)$.

We can summarize the last three propositions:

THEOREM 2.15. (i) *A Euclidean circle is a spherical circle iff it is a non-Euclidean circle* (*provided it lies in* Δ^-).

(ii) *A spherical circle is a Euclidean circle iff it is a non-Euclidean circle* (*provided it lies in* Δ^-).

(iii) *A Euclidean circle is a spherical circle iff its Euclidean center is a scalar multiple of a unitary matrix.*

In each of the above cases, the spherical circle has a blockwise a-unitary matrix for its center, and the non-Euclidean circle has a blockwise c-$\mathscr{J}$-unitary matrix for its center.

One may verify in a similar way that the Euclidean lines correspond to the spherical circles which pass through the pole $(I\ O)$, and the Euclidean lines through the origin correspond to the spherical circles which pass through $(I\ O)$ and $(O\ I)$, and have $1/\sqrt{2}$ as radius.

In [3] we have seen that using Möbius transforms, one can verify that every pair of points $\mathbf{P}$ and $\mathbf{Q}$ on K lies on a great circle. We suggest here another approach to verify this property, that has some interest in its own right from the point of view of properties of unitary matrices.

LEMMA 2.16. *Let* $\mathscr{U}$ *be a* $2n \times 2n$ *unitary matrix. There exist two unitary* *matrices $\mathscr{V}$, $\mathscr{W}$ such that $\mathscr{U} = \mathscr{V}^*\mathscr{W}$, and $\mathscr{V}$, $\mathscr{W}$ are $1/\sqrt{2}$-blockwise unitary matrices.*

Proof. By [3], it will suffice to prove the lemma for $\mathscr{U} = \begin{pmatrix} L_1 & L_2 \\ -L_2 & L_1 \end{pmatrix}$, where $L_1^2 + L_2^2 = I$, and L_1, L_2 are real nonnegative diagonal matrices. Let $\mathscr{V}^* = 1/\sqrt{2}\begin{pmatrix} I & iI \\ -iI & I \end{pmatrix}$, where I is the $n \times n$ identity matrix. Then $\mathscr{V}$ is a $1/\sqrt{2}$-blockwise unitary matrix. Set $\mathscr{W} = \mathscr{V}\mathscr{U}$. Then $\mathscr{W}$ is a unitary matrix, and $\mathscr{U} = \mathscr{V}^*\mathscr{W}$. To complete the proof, one needs to verify by straightforward computations that $\mathscr{W} = 1/\sqrt{2}\begin{pmatrix} L_1 + iL_2 & L_2 - iL_1 \\ -iL_1 - L_2 & -iL_2 + L_1 \end{pmatrix}$ is a $1/\sqrt{2}$-blockwise unitary matrix, whenever $L_1^2 + L_2^2 = I$. ∎

As a result we may derive

THEOREM 2.17. *Let* **P** *and* **Q** *be points on K. Then there exists a great circle (of radius $1/\sqrt{2}$) on which both* **P** *and* **Q** *lie.*

Proof. Let $\mathscr{P}$ and $\mathscr{Q}$ be unitary completions of **P** and **Q**, and set $\mathscr{U} = \mathscr{P}\mathscr{Q}^*$. Then $\mathscr{U}$ is a unitary matrix, and by Lemma 2.16, there exist $1/\sqrt{2}$-blockwise unitary matrices $\mathscr{V}$, $\mathscr{W}$ so that $\mathscr{U} = \mathscr{V}^*\mathscr{W}$. Consequently $\mathscr{V}^*\mathscr{W} = \mathscr{P}\mathscr{Q}^*$ or $\mathscr{V}\mathscr{P} = \mathscr{W}\mathscr{Q} = \mathscr{R}$, and $\mathscr{R}$ is a unitary matrix. Let **R** be a point on K, for which $\mathscr{R}$ is a unitary completion.

Then, on the great circle of radius $1/\sqrt{2}$, centered at **R**, both **P** and **Q** lie. To verify this, note that $\mathscr{P}\mathscr{R}^*$ and $\mathscr{Q}\mathscr{R}^*$ are $1/\sqrt{2}$-blockwise unitary matrices. ∎

To complete this topic, one is tempted to show that given two points **P** and **Q** on K, that are not antipodal points, there exists precisely one circle on which both **P** and **Q** lie. The following example shows that such a statement is false.

EXAMPLE. Let $U(\lambda) = \begin{pmatrix} \lambda I & (1-\lambda^2)^{1/2} I \\ -(1-\lambda^2)^{1/2} I & \lambda I \end{pmatrix}$, where I is the $n \times n$ identity matrix, for $n > 1$, and λ, $0 < \lambda < 1$, a real number. $U(\lambda)$ is a unitary matrix. Let E be a $n \times n$ diagonal matrix, whose diagonal elements are either i or $-i$. Let $\mathscr{V}(E) = 1/\sqrt{2}\begin{pmatrix} E & I \\ I & E \end{pmatrix}$. Then $\mathscr{V}(E)$ is a $1/\sqrt{2}$-blockwise unitary matrix, and $\mathscr{W}(E, \lambda) = \mathscr{V}(E)\,\mathscr{U}(\lambda)^*$ is a unitary matrix for each matrix E and real number λ, $0 < \lambda < 1$. A straightforward computation shows that $\mathscr{W}(E, \lambda)$ is a $1/\sqrt{2}$-blockwise unitary matrix.

In terms of radius and circles, this proves that the set $\mathscr{L}$ of all points $(\lambda I \;\; (1-\lambda^2)^{1/2} I)$, for all λ, $0 < \lambda < 1$, lies on circles of radius $1/\sqrt{2}$ centered at $(1/\sqrt{2}E \;\; 1/\sqrt{2}I)$ for any $n \times n$ diagonal matrix E with diagonal entries $+i$ or $-i$. For $n > 1$ one gets in this way different centers, that are not antipodal of each other. In particular this implies that there are different circles of radius $1/\sqrt{2}$ that contain all the points of $\mathscr{L}$.

By interchanging roles, we get that the different centers lie on infinitely many circles of radius $1/\sqrt{2}$, namely, all those circles of radius $1/\sqrt{2}$, centered at any point of $\mathscr{L}$.

3. The Riemann Sphere and the Stereographic Projection

The set of $n \times n$ complex matrices is denoted by $M_n(\mathbb{C})$, and the subset of the Hermitian matrices is denoted by $H_n(\mathbb{C})$. We consider the space $\mathscr{S} = M_n(\mathbb{C}) \times H_n(\mathbb{C})$ (for $n=1$ this reduces to $\mathbb{C} \times R$ which is naturally equivalent to R^3).

For each point $(P \;\; H) \in \mathscr{S}$ we set $|||(P \;\; H)||| = \| PP^* + HH^* \|^{1/2}$, where the right-hand norm is the usual spectral norm in $M_n(\mathbb{C})$. The so defined (generalized) norm $||| \;\; |||$ in $\mathscr{S}$ induces a metric which is equivalent to the product metric of $H_n(\mathbb{C}) \times H_n(\mathbb{C}) \times H_n(\mathbb{C})$.

We shall denote the distance between two points of $\mathscr{S}$ induced by this metric by $d((P_1 \;\; H_1), (P_2 \;\; H_2))$.

The set of finite points P_1 in the complex matrix projective line $P_1(M_n(\mathbb{C}))$, is naturally isomorphic to the Euclidean plane $M_n(\mathbb{C})$. The isomorphism given by $i_0(P_1 \;\; P_2) = P_2^{-1} P_1$ preserves the Euclidean distance. In particular, if we identify $P_2^{-1} P_1$ with $i(P_1 \;\; P_2) = (P_2^{-1} \;\; P_1 I) \in P_1(M_n(\mathbb{C}))$, the set $(V \;\; I) \in P_1(M_n(\mathbb{C}))$ is naturally isomorphic to the Euclidean plane. The map e: $P_1 \to M_n(\mathbb{C}) \times \{0\} \subset \mathscr{S}$, given by $e(V \;\; I) = (V \;\; 0)$ is a distance preserving isomorphism of P_1 onto $M_n(\mathbb{C}) \times \{0\}$.

Recall that each point, $(P_1 \;\; P_2) \in P_1(M_n(\mathbb{C}))$ has a representative on $\mathscr{K}$ $(\mathscr{K} = \{(P_1 \;\; P_2) \mid P_1 \; P_1^* + P_2 \; P_2^* = I\})$ and $(P_1 \;\; P_2) = U(\bar{P}_1 \;\; \bar{P}_2)$ for $(P_1 \;\; P_2)$, $(\bar{P}_1 \;\; \bar{P}_2) \in \mathscr{K}$ iff U is a unitary matrix. This happens whenever the two points represent the same point in $P_1(M_n(\mathbb{C}))$.

Let

$$\mathscr{R} = \{(P \;\; H) \mid (P \;\; H) \in M_n(\mathbb{C}) \times H_n(\mathbb{C}), PP^* + HH^* = H\}. \tag{3.1}$$

This set $\mathscr{R}$ consists of all the points on the sphere with radius $\frac{1}{2}$, whose center is located at $(O \;\; \frac{1}{2} I)$. We refer to $\mathscr{R}$ *as the (generalized) Riemann sphere.*

Define a map

$$s: P_1(M_n(\mathbb{C})) \to \mathscr{R}, \qquad s(P_1 \;\; P_2) = (P_2^* P_1 \;\; P_2^* P_2) \qquad \text{for} \quad (P_1 \;\; P_2) \in K. \tag{3.2}$$

Since $(P_1 \;\; P_2)$ is determined up to a left multiplication by a unitary matrix, s is well defined. That $s(P_1 \;\; P_2)$ is actually in $\mathscr{R}$ follows by straightforward computation, and the fact that $(P_1 \;\; P_2) \in \mathscr{K}$.

For every point $(P\ H)$, $(P\ H) \in \mathscr{R}$, the relation $PP^* + HH^* = H$ implies that H is an Hermitian, semi-positive definite matrix. It follows that $PP^* = H(I-H)$. In particular, $H = UL^2U^*$, for some unitary matrix U, and a real diagonal matrix L. Choose $P_1 = U(I-L^2)^{1/2}V$ and $P_2 = ULU^*$, for a suitable unitary matrix V, so that $P_2^*P_1 = P$, $P_2^*P_2 = H$. Therefore, $s(P_1\ P_2) = (P\ H)$. Hence

LEMMA 3.1. *The map* s: $P_1(M_n(\mathbb{C})) \to \mathscr{R}$, *is an epimorphism.*

The map s is in general not a one-to-one map. However,

LEMMA 3.2. *The map* s: $P_1(M_n(\mathbb{C})) \to \mathscr{R}$, *induces a one-to-one map on* P_1.

Proof. Let $(P_1\ P_2)$, $(\bar{P}_1\ \bar{P}_2) \in K$, $|P_2| \neq 0$, and let $s(P_1\ P_2) = s(\bar{P}_1\ \bar{P}_2)$. Then $P_2^*P_2 = \bar{P}_2^*\bar{P}_2$, or $P_2^* = \bar{P}_2^*U^*$, for some unitary matrix U. Also $P_2^*P_1 = \bar{P}_2^*\bar{P}_1$, that is $P_2^*P_1 = P_2^*U\bar{P}_1$, and since $|P_2| \neq 0$, we have $P_1 = U\bar{P}_1$. Whence $(P_1\ P_2) = U(\bar{P}_1\ \bar{P}_2)$, and these points represent the same point. ∎

Note that we have proved that in fact, if $(P\ H) \in \mathscr{R}$, and $|H| \neq 0$, then $s^{-1}(P\ H)$ is a single point in $P_1(M_n(\mathbb{C}))$.

For a pair of points $\mathbf{P} = (P_1\ P_2)$, $\mathbf{Q} = (Q_1\ Q_2)$, so that $\mathbf{P}, \mathbf{Q} \in \mathscr{K}$, we denote by $\mathscr{P}$, $\mathscr{Q}$, and $\mathscr{U}$ the unitary matrices:

$$\mathscr{P} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}, \qquad \mathscr{Q} = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}, \qquad \mathscr{U} = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} = \mathscr{P}\mathscr{Q}^*. \tag{3.3}$$

Recall that the chordal distance $\chi(\mathbf{P}, \mathbf{Q})$ is given by $\chi(\mathbf{P}, \mathbf{Q}) = \| U_2 \|$, and $\chi(\mathbf{P}, \hat{\mathbf{Q}}) = \| U_1 \|$, where $\hat{\mathbf{Q}} \in \mathscr{K}$ is the antipodal point to $\mathbf{Q}$, also $\hat{\mathbf{Q}} = (Q_3\ Q_4)$.

For a point $(P\ H) \in \mathscr{R}$, the point $(-P, I-H)$ is its antipodal point. It follows that

$$s(\hat{\mathbf{Q}}) = \text{antipodal point on } \mathscr{R} \text{ to } s(\mathbf{Q}). \tag{3.4}$$

We also have

$$s(\mathbf{Q}) = (Q_2^*\quad Q_4^*)\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\mathscr{Q}, \tag{3.5}$$

$$s(\hat{\mathbf{Q}}) = (Q_2^*\quad Q_4^*)\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\mathscr{Q}. \tag{3.6}$$

As for the distance $\chi(\mathbf{PQ})$ and $d(s(\mathbf{P}), s(\mathbf{Q}))$ we have

LEMMA 3.3. *$d(s(\mathbf{P}), s(\mathbf{Q})) \leqslant \chi(\mathbf{P}, \mathbf{Q})$. Equality holds whenever $\mathscr{U}$ is a blockwise r-unitary matrix.*

Proof. Since $\mathscr{P} = \mathscr{U}\mathscr{Q}$ and $\|(Q_2^* \;\; Q_4^*)\| = 1$, we have

$$d^2(s(\mathbf{P}), s(\mathbf{Q})) = \left|\left|\left| (Q_2^* \;\; Q_4^*) \begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix} (U_1 \;\; U_2)\mathscr{Q} - (Q_2^* \;\; Q_4^*) \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \mathscr{Q} \right|\right|\right|$$

$$= \left|\left|\left| (Q_2^* \;\; Q_4^*) \begin{pmatrix} U_1^* U_1 - I & U_1^* U_2 \\ U_2^* U_1 & U_2^* U_2 \end{pmatrix} \right|\right|\right|,$$

but,

$$\begin{pmatrix} U_1^* U_1 - I & U_1^* U_2 \\ U_2^* U_1 & U_2^* U_2 \end{pmatrix} \begin{pmatrix} U_1^* U_1 - I & U_1^* U_2 \\ U_2^* U_1 & U_2^* U_2 \end{pmatrix} = \begin{pmatrix} U_3^* U_3 & 0 \\ 0 & U_2^* U_2 \end{pmatrix},$$

whence, $d^2(s(\mathbf{P}), s(\mathbf{Q})) \leqslant \| U_3 \| = \| U_2 \| = \chi(\mathbf{P}, \mathbf{Q})$.

Furthermore, if $\mathscr{U}$ is a blockwise r-unitary matrix, then $U_3^* U_3 = U_2^* U_2 = (1 - r^2)I$, and consequently equality holds as stated. ∎

In particular, we may derive

PROPOSITION 3.4. *The image of a circle $\gamma_s(\mathbf{M}, r)$ in $P_1(M_n(\mathbb{C}))$, is a circle of radius r in $\mathscr{R}$, centered at $s(M)$.*

Observe that the set of points at distance r from $(P \;\; H)$ have distance $(1 - r^2)^{1/2}$ from $(-P, I - H)$. This is obviously the case with the points of $\gamma_s(M, r)$, by Lemma 3.3.

We turn now to the converse: Given a circle γ_e on $\mathscr{R}$, is there a circle γ_s in $P_1(M_n(\mathbb{C}))$ so that $s(\gamma_s) = \gamma_e$? To answer it, we need

LEMMA 3.5. *Let $(Q_1, Q_2) \in \mathscr{K}$ be a point so that $s(Q_1 \;\; Q_2) = (Q \;\; H)$. Let $(P \;\; L) \in \mathscr{R}$ be a point so that $d^2((P \;\; L), \;\; (Q \;\; H)) + d^2((P \;\; L), (-Q \;\; I - H)) = 1$. Then there exist unitary matrices $\bar{U}_1$, $\bar{U}_2$, and a real non-negative number l, so that $0 \leqslant l \leqslant 1$, and $s((l\bar{U}_1 \;\; (1 - l^2)^{1/2} \bar{U}_2)Q) = (P \;\; L)$.*

Proof. Let $(P_1 \;\; P_2)$ be any point in $\mathscr{K}$ so that $s(P_1 \;\; P_2) = (P \;\; L)$. For some unitary matrix $\mathscr{U} = \left(\begin{smallmatrix} U_1 & U_2 \\ U_3 & U_4 \end{smallmatrix}\right)$ we have $(P_1 \;\; P_2) = (U_1 \;\; U_2)\mathscr{Q}$. Let

$$D_1 = (Q_2^* \;\; Q_4^*) \begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix} (U_1 \;\; U_2)\mathscr{Q} - (Q_2^* \;\; Q_4^*) \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \mathscr{Q}, \tag{3.7}$$

$$D_2 = (Q_2^* \;\; Q_4^*) \begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix} (U_1 \;\; U_2)\mathscr{Q} - (Q_2^* \;\; Q_4^*) \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \mathscr{Q}. \tag{3.8}$$

Then

$$|||D_1|||^2 + |||D_2|||^2 = 1 \text{ and } D_1 D_1^* + D_2 D_2^* = I, \tag{3.9}$$

$$D_1 D_1^* = (Q_2^* \quad Q_4^*) \begin{pmatrix} U_3^* U_3 & 0 \\ 0 & U_2^* U_2 \end{pmatrix} \begin{pmatrix} Q_2 \\ Q_4 \end{pmatrix}, \tag{3.10}$$

$$D_2 D_2^* = (Q_2^* \quad Q_4^*) \begin{pmatrix} U_1^* U_1 & 0 \\ 0 & U_4^* U_4 \end{pmatrix} \begin{pmatrix} Q_2 \\ Q_4 \end{pmatrix}. \tag{3.11}$$

For suitable diagonal matrices L', L'' so that $(L')^2 + (L'')^2 = I$, and for V_1, V_2, V_3, V_4 unitary matrices we have:

$$U_1 = V_1 L' V_2, \qquad U_2 = V_1 L'' V_3, \qquad U_3 = -V_4 L'' V_2, \qquad U_4 = V_4 L' V_3. \tag{3.12}$$

Denote

$$W_1 = V_1 V_2, \qquad W_2 = V_1 V_3, \qquad W_3 = V_4 V_2, \qquad W_4 = V_4 V_3. \tag{3.13}$$

Then from (3.9), (3.10), and (3.11) we have

$$Q_2^* U_1^* = l Q_2^* W_1, \qquad Q_2^* U_3^* = (1-l^2)^{1/2} Q_2^* W_3, \tag{3.14}$$

$$Q_4^* U_2^* = (1-l^2)^{1/2} Q_4^* W_2, \qquad Q_4^* U_4^* = l Q_4^* W_4, \tag{3.15}$$

where $0 \leqslant l = \| U_1 \| \leqslant 1$, $0 \leqslant (1-l^2)^{1/2} = \| U_2 \| \leqslant 1$.

From (3.14) and (3.12) it follows that

$$Q_2^* U_1^* U_1 = l^2 Q_2^* W_1^* W_1, \qquad Q_2^* U_1^* U_2 = l(1-l^2)^{1/2} Q_2^* W_1^* W_2. \tag{3.16}$$

From (3.15) and (3.12) it follows that

$$Q_4^* U_2^* U_1 = l(1-l^2)^{1/2} W_2^* W_1, \qquad Q_4^* U_2^* U_2 = (1-l^2) Q_4^* W_2^* W_2. \tag{3.17}$$

In particular, for $W_1 = \bar{U}_1$ and $W_2 = \bar{U}_2$ we have

$$s((l\bar{U}_1 \quad (1-l^2)^{1/2} \bar{U}_2) \mathscr{Q}) = s((U_1 \quad U_2) \mathscr{Q} = (P \ L)$$

as stated. ∎

We may summarize the properties of the map s as follows:

THEOREM 3.6. *The map s: $P_1(M_n(\mathbb{C})) \to \mathscr{R}$ induces an isomorphism on P_1. It carries circles into circles. For every circle γ_e in $\mathscr{R}$, the set $s^{-1}(\gamma_e)$ contains a circle γ_s so that $s(\gamma_s) = \gamma_e$. The radius of γ_e and γ_s are the same, and the center of γ_e is the image of the center of γ_s. The map s is a continuous, distance decreasing (in the weak sense) epimorphism.*

Note that the points of $P_1(M_n(\mathbb{C}))$ have no single representatives on $\mathscr{K}$, but we can find natural representatives for points in P_1. A point in P_1, $(P_1\ P_2)$ has a single representative on $\mathscr{K}$, under the restriction $P_2 = P_2^* > 0$.

Define a map α: $P_1(M_n(\mathbb{C})) \to P_1(M_n(\mathbb{C}))$, by $\alpha(\mathbf{P}) = \hat{\mathbf{P}}$, where $\hat{\mathbf{P}}$ is the antipodal point of $\mathbf{P}$ (we always consider representatives for the points in $P_1(M_n(\mathbb{C}))$ taken in $\mathscr{K}$). One readily verifies that $\alpha(\gamma_s(\hat{\mathbf{M}}, r)) = \gamma_s(\mathbf{M}, r)$, hence α preserves circles. Also $\alpha^2 = \mathrm{Id}$, hence α^{-1} of a circle, is a circle.

For a unitary matrix $\mathscr{M}$, let us denote by $\mu(\mathbf{P})$ the induced Möbius transformation, that is $\mu(\mathbf{P}) = \mathbf{P}\mathscr{M}$. The map μ is a (chordal) distance preserving isomorphism. In particular, it carries circles into circles, and μ^{-1} of a circle is a circle.

Since $\mathscr{P}\mathscr{M}$ is a unitary matrix, where $\mathscr{P} = \binom{\mathbf{P}}{\hat{\mathbf{P}}}$, it follows that $\alpha\mu(\mathbf{P}) = \mu\alpha(\mathbf{P})$.

Furthermore, since for a unitary matrix $\mathscr{U} = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$ we have $\| U_2 \| = \| U_3 \|$, it follows that α is (chordal) distance preserving.

Summarizing, we have

LEMMA 3.7. *The map* α: $P_1(M_n(\mathbb{C})) \to P_1(M_n(\mathbb{C}))$ *is a distance preserving isomorphism, that commutes with every distant preserving Möbius transformation.*

Note, in particular, that the Möbius transformation ν, induced by the unitary matrix $\mathscr{J}$, represents a distance preserving isomorphism that commutes with α.

Let σ denote the embedding σ: $M_n(\mathbb{C}) \to P_1(M_n(\mathbb{C}))$ by $\sigma(V) = (V\ I)$. The map σ is a (Euclidean) distance preserving isomorphism of $M_n(\mathbb{C})$ and P_1. In particular, circles are mapped onto circles. By the results of Section 2 a circle is mapped onto a (spherical) circle in $P_1(M_n(\mathbb{C}))$ iff the center V satisfies $V = \lambda V_0$, where $V_0 V_0^* = I$.

In fact, $i_0 \sigma = Id$: $M_n(\mathbb{C}) \to M_n(\mathbb{C})$.

To define the sterographic projection π from $\mathscr{R}$ onto $M_n(\mathbb{C}) \times \{0\}$, we use composition of previously studied maps. This will enable us to deduce that π carries circles into circles.

We start by defining the inverse map $⅄$ of π, which is more natural, and easier to follow:

$$⅄: M_n(\mathbb{C}) \times \{0\} \xrightarrow{e^{-1}} P_1 \subset P_1(M_n(C)) \xrightarrow{\alpha} P_1(M_n(C)) \xrightarrow{\nu} P_1(M_n(C)) \xrightarrow{s} \mathscr{R}. \tag{3.18}$$

One can derive the following formulae:

$$⅄(V\ 0) = (V(I + V^*V)^{-1} \quad V(I + V^*V)^{-1} V^*) \tag{3.19}$$

or

$$\amalg(V\ 0) = (V(I + V^*V)^{-1} \quad (I + VV^*)^{-1}\, VV^*), \tag{3.19'}$$

In particular, if $\amalg(V\ 0) = (Q\ H)$, then for $V = 0$, $\|Q\| = \|H\| = 0$, and for $V \neq 0$, $0 < \|H\| < 1$ and $0 < \|Q\| < 1$.

Conversely, if $0 < \|Q\| < 1$, then necessarily $|H| \neq 0$, and $s^{-1}(Q\ H)$ is uniquely determined. Furthermore, if $s^{-1}(Q\ H) = (P_1\ P_2)$ then $|P_2| \neq 0$. If $0 < \|H\| < 1$ then $|P_1| \neq 0$. Consequently, $\alpha^{-1}v^{-1}s^{-1}(Q\ H) \in P_1$, and there exists a point $(V\ 0)$ in $M_n(\mathbb{C}) \times \{0\}$ for which $\amalg(V\ 0) = (Q\ H)$. The point $(V\ 0)$ is uniquely determined, whence we can define $\pi(Q\ H) = (V\ 0)$. One can derive the formulae

$$\pi(Q\ H) = ((I - H)^{-1}\, Q\ 0) \tag{3.20}$$

$$(I - H)^{-1} = I + VV^* \tag{3.21}$$

which are well defined whenever $\|H\| < 1$.

One verifies that although s^{-1} is not always uniquely determined, π is well defined. Furthermore

$$\amalg\pi(Q\ H) = (Q\ H) \qquad \text{whenever} \quad \|H\| < 1, \text{and } (Q, H) \in \mathscr{R},$$

$$\pi\amalg(V\ 0) = (V\ 0) \qquad \text{for all} \quad V \in M_n(\mathbb{C}).$$

We can summarize these results as:

THEOREM 3.8. *The stereographic map* $\pi(Q\ H) = ((I - H)^{-1}Q\ 0)$, *defined on* $\{(Q\ H) \mid (Q\ H) \in \mathscr{R},\ \|H\| < 1\}$ *is a well-defined map, whose inverse from* $M_n(\mathbb{C}) \times \{0\}$ *is* $\amalg$. *The map* π *carries circles on* $\mathscr{R}$ *to circles in* $M_n(\mathbb{C})$ *(and vice versa) whenever their center in* $\mathscr{R}$ *is* (cQ_0, dI), *where c and d are scalars and* $Q_0Q_0^* = I$, *(whenever their center in* $M_n(C) \times \{0\}$ *is of the form* $(\lambda V_0\ 0)$, λ *a scalar,* $V_0V_0^* = I$).

Notice that using the results of Section 2a we can derive the relations between centers and radii via (3.19′) and (3.20).

One may observe that the Hermitian lines in $M_n(C) \times \{0\}$ correspond to circles on $\mathscr{R}$ that pass through $(0\ I)$, and lines through the origin correspond to great circles on $\mathscr{R}$ that pass through $(0\ I)$, and thus necessarily also through $(0\ 0)$.

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